1 Examples of probability distributions

1.1 Bernoulli, Binomial and multinomial distributions

Suppose we have a basic experiment which can result in either a "success" or a "failure". Let p the probability of a success and $q = 1 - p$ the probability of a failure. This is known as a Bernoulli trial. The mass function can be expressed as

$$f(x) = p^x (1-p)^{1-x}, x = 0, 1$$

The mean and variance are

$$\mu = p.\sigma^2 = p(1-p)$$

As an example, suppose we toss a balanced die once. Identify the event "success" with getting a "3". Then $p = 1/6, q = 5/6$.

Example Throw a balanced die 6 times. What is the probability of observing exactly 3 ones?

Here, $p = \frac{1}{6}$. Hence, $P(X = 3) = \binom{6}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3 = \frac{625}{11664} = 0.054$

Example We have 5 switches. The probability that a switch fails is 0.1. Then

$$\begin{aligned} P(\text{at most one fails}) &= P(X \leq 1) = 0.9185 \\ P(\text{none fails}) &= P(X = 01) = 0.5905 \end{aligned}$$

The binomial distribution arises in a natural way in several repetitions of a basic Bernoulli experiment. Suppose we repeat a Bernoulli experiment n times under independent and identical conditions. Let X be a random variable which counts the number of successes. Then X has a binomial distribution given by

$$f\left(x\right)=\binom{n}{x}p^{x}\left(1-p\right)^{n-x},x=0,1,...,n$$

The mean and variance are

$$\mu=np.\sigma^{2}=np\left(1-p\right)$$

The multinomial distribution generalizes the binomial. Suppose we have a basic experiemt that can result in one of k possible outcomes with probabilities $p_1, ..., p_k$ respectively, $\sum p_i = 1$. Suppose that we repeat this experiment n times under i.i.d. conditions and we observe frequencies $x_1, ..., x_k$ of occurrence respectively for the possible outcomes, with $\sum x_i = n$. Then, the probability distribution is given by

$$f(x_1, ..., x_k) = \frac{n!}{x_1!...x_k!} p_1^{x_1} ... p_k^{x_k}$$

Covariance $(X_1, X_2) = -np_1p_2$

Example Toss a balanced die twice. The probability of observing exactly one "5" and one "6" is

$$\frac{2!}{1!1!} \left(\frac{1}{6}\right)^2 = \frac{1}{18}$$

1.2 Hypergeometric distribution

The Hypergeometric distribution is given by

$$h\left(x; N, n, k\right) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}, x \leq k, n - x \leq N - k$$

This is the distribution we have for sampling without replacement n items from a box containing k items of one kind and $N - k$ items of another kind. The variable X records the number of items of the first kind.

The mean and variance of the random variable X are respectively

$$\mu = \frac{nk}{N}, \sigma^2 = \left(\frac{N-n}{N-1}\right) n \frac{k}{N}\left(1 - \frac{k}{N}\right)$$

Note, for $\frac{n}{N} \leq 0.05$, we can use the binomial approximation.

Example Suppose that i phones come in lots of 10. We inspect them and accept the entire lot if in a random sample of 3 without replacement, none are defective. If the lot contains 2 defective i phones the probability of accepting the lot is

$$P(X=0) = \frac{\binom{2}{0}\binom{8}{3}}{\binom{10}{3}} = 0.467$$

Hence, approximately 47% of the time, we will accept such lots. Equivalently, we will reject such lots approximately 53% of the time. We can improve on this probability if we take a larger sample. For example,

$$P(X=0) \quad = \begin{cases} 0.3 & n=4 \\ 0.22 & n=5 \end{cases}$$

1.3 Negative binomial and geometric

Suppose that a Bernoulli experiment is repeated until we observe the first success. Let X denote the number of repetitions needed. Then X has a geometric distribution given by

$$f(x) = p(1-p)^{x-1}, x = 1, 2, ...$$

The mean and variance of the random variable X are respectively

$$\mu = \frac{1}{p}, \sigma^2 = \frac{1-p}{p^2}$$

Note, $P(X > x) = \sum_{k=x+1}^{\infty} p(1-p)^{k-1} = (1-p)^x$

We may generalize the geometric distribution by looking at the number of repetitions needed to obtain the k^{th} success.

$$f(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}, x = k, k+1, ...$$

The mean and variance in that case are respectively

$$\mu = \frac{k}{p}, \sigma^2 = k\left(\frac{1-p}{p^2}\right)$$

Example A door is closed repeatedly to test it for wear. The probability that it malfunctions at any one closings is $p = 0.001$. What is the probability that it will first malfunction after 100 closings?

$$0.001\,(1 - 0.001)^{99} = 9.057 \times 10^{-4}$$

Note that

$$\begin{aligned} P\,(X \leq x) &= 1 - P\,(X > x) = 1 - (1 - p)^{x} \\ &= \begin{cases} 0.0943 & x = 99 \\ 0.63194 & x = 999 \\ 1 & x = 9999 \end{cases} \end{aligned}$$

1.4 Poisson distribution and the Poisson process

Suppose that an individual receives calls on his cellular phone during a time interval $(0, t)$ .What is the probability that he will receive exactly 5 calls during that time interval?

In order to answer this question, we need to develop a model. We begin by dividing the interval into n equal segments of length $\frac{t}{n}$ each. We make the following assumptions:

i) The probability of receiving a call in a segment is proportional to the length of the segment i.e. $p = \lambda\left(\frac{t}{n}\right)$

ii) The events in different intervals are independent

iii) When $n \to \infty, p \to 0$, np remains constant.

Under those assumptions, the random variable X which counts the number of calls received follows a binomial distribution with parameters n, p

Theorem Let X be a binomial random variable with probability distribution $b(x; n, p)$. Then when $n \to \infty, p \to 0$ and np remains constant

$$b(x; n, p) \to p(x; \mu) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, ...$$

The mean and variance of the Poisson distribution are respectively

$$\mu = \lambda t, \sigma^2 = \lambda t$$

This theorem can also be used to approximate binomial probabilities. The approximation is good if $n \geq 20, p \leq 0.05$ or if $n \geq 100, p \leq 0.10$

Example Suppose we have a binomial distribution with parameters $n = 400, p = 0.005$,$np = 2$. Then $P(X = 1) = F(1) - F(0) = 0.270671$. The exact value is 0.270669.

The Poisson approximation to the binomial is usually good when $n \geq 20, p \leq 0.05$ or if $n \geq 100, p \leq 0.10$

Example Suppose that on average an individual receives 5 calls in 10 minutes. What is the probability that he receives exactly 2 calls in one minute?

Assuming we have a Poisson process, we know that the mean $\lambda t = 5$. Substituting $t = 10$, we see that $\lambda = \frac{5}{10}$. The time interval is now changing to one minute where the mean is $\lambda t = \frac{5}{10}(1)$.

$$P(X = 2) = \frac{e^{-\frac{1}{2}}}{2!}\left(\frac{1}{2}\right)^2$$

TableA.2p.434 For $x = 2$, the probability is $0.9856 - 0.9098 - 0.0758$.

1.5 Continuous uniform

The mean and variance of the uniform distribution on the interval (A, B) are

$$\mu = \frac{A+B}{2}, \sigma^2 = \frac{(B-A)^2}{12}$$

Example Pick a number at random from the interval $(0, 1)$ Repeat this 5 times. What is the probability that at most 2 of the numbers will be leass than 0.25?

This is a two part problem involving the uniform and the binomial. In the first part, we calculate the probability of having a number less than 0.25. It is from the uniform equal to $p = 0.25$. Now we apply the binomial with $n = 5, p = 0.25$. The probability of at most 2 is 0.8965 from Table A.1 p.428.

1.6 Normal distribution

The normal distribution is by far the most important distribution in probability and statistics. It has acquired its distinction becasue of the central limit theorem which states that the limiting distribution of the mean of a random sample from a distribution having finite variance is normal .The density has the following form

$$f(x;\mu,\sigma)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}},-\infty<x<\infty,-\infty<\mu<\infty,\sigma>0$$

The mean and variance of the normal distribution are respectively

$$\mu,\sigma^2$$

The special case $\mu=0,\sigma=1$ defines the standard normal distribution whose cdf is denoted by $\Phi(x)$.

TableA.3p436 provides the area under the curve between any two points for the standard normal distribution.

The following theorem shows that only the standard normal needs to be tabulated.

Theorem Let X be a random variable with distribution $n(x; \mu, \sigma)$. Then the transformed variable $Z = \frac{X-\mu}{\sigma}$ haas a standard normal distribution.

As a consequence, we can compute

$$P(x_1 < X < x_2) = \Phi\left(\frac{x_2 - \mu}{\sigma}\right) - \Phi\left(\frac{x_1 - \mu}{\sigma}\right)$$

Note: i) $P(X \geq x) = 1 - \Phi(x)$

ii) $\Phi(-x) = 1 - \Phi(x)$

iii) $\Phi(-1.96) = 0.025$; $\Phi(1.96) = 0.9750$

Example Find x such that $\Phi(x) = 0.3$.

From Table A.3, $\Phi(-0.52) = 0.3015, \Phi(-0.54) = 0.2981$. Using linear interpolation,

$$\frac{x - (-0.52)}{0.30 - (0.3015)} = \frac{-0.52 - (-0.54)}{0.3015 - (0.2981)}$$

we find $x = -0.5288$

Example1 For a $n\left(x;3,0.005\right)$,

$$P\left(X<2.99\right)=0.0228=P\left(X>3.01\right)$$

Example2 For a $n\left(x;1.5,0.2\right)$,

$$P\left(X<1.108\right)=0.025=P\left(X>1.892\right)$$

Example For a $n\left(x;10,5\right)$ find

$$\begin{aligned}P\left(5<X<8\right) &= \Phi\left(\frac{8-10}{5}\right)-\Phi\left(\frac{5-10}{5}\right)\\ &= \Phi\left(-0.4\right)-\Phi\left(-1.0\right)=0.3446-0.1587=0.1859\end{aligned}$$

Example Specifications for the diameter of ball bearings are 3 ± 0.01cm. Assuming that the diameter follows a normal distribution with mean 3 and variance 0.005^2, what proportion falls within the specifications?

We need to calculate

$$\begin{aligned}P\left(3-0.01<X<3+0.01\right) &= \Phi\left(\frac{3.01-3}{0.005}\right)-\Phi\left(\frac{3.0-0.01-3}{0.005}\right)\\ &= \Phi\left(2\right)-\Phi\left(-2\right)=0.9544\end{aligned}$$

Hence, about 4.6% fall outside the specifications.

Theorem Let X be a binomial random variable with mean and variance $\mu = np, \sigma^2 = np(1-p)$. Then, as $n \to \infty$,the distribution of

$$Z = \frac{X - np}{\sqrt{npq}}$$

is that of a standard normal.

This theorem enables us to approximate binomial probabilities with those of a normal.

Example Suppose that we have a binomial distribution with parameters $n = 50, p = 0.05$. Compute $P(X = 4)$Since $np = 2.5$

$$\begin{aligned} P(X = 4) &\simeq \Phi\left(\frac{4.5 - 2.5}{\sqrt{2.5(1 - 0.05)}}\right) - \Phi\left(\frac{3.5 - 2.5}{\sqrt{2.5(1 - 0.05)}}\right) \\ &= \Phi(1.2978) - \Phi(0.64889) \\ &= 0.90282 - 0.741795 = 0.161 \end{aligned}$$

The exact value is $0.8964 - 0.7604 = 0.136$.

Example Suppose that we have a binomial distribution with parameters $n = 100, p = 0.05$. Compute $P(X = 10)$Since $np = 5$,$np(1-p) = 4.75$

$$\begin{aligned} P(X = 10) &\simeq \Phi\left(\frac{10.5-5}{\sqrt{4.75}}\right) - \Phi\left(\frac{9.5-5}{\sqrt{4.75}}\right) \\ &= \Phi(2.52) - \Phi(2.06) \\ &= 0.9941 - 0.9803 = 0.0138 \end{aligned}$$

Also, $P(5 \leq X \leq 10) = P(4.5 \leq X \leq 10.5) = 0.58471$

Table 1: Normal approximation to the binomial

Table 3.1 P.141 shows how well the approximation works. It is usually good whenever $np \geq 5$ and $n(1-p) \geq 5$

1.7 Gamma and exponential distribution

We first derive the exponential distribution.

Returning to the derivation of the Poisson distribution, let T be the time until the first call received by an operator. Then,

$$\begin{aligned} P(T > t) &= P(\text{"}0\text{"} \, successes) \\ &= \tbinom{n}{0} (1-p)^n \\ &= \left(1 - \frac{\lambda t}{n}\right)^n \to e^{-\lambda t} \end{aligned}$$

Hence, $F(t) = P(T \leq t) = 1 - e^{-\lambda t}$ and the density becomes $f(x) = \lambda e^{-\lambda x}, x > 0, \lambda > 0$.

The mean and variance of the exponential distribution are

$$\mu = \lambda^{-1}, \sigma^2 = \lambda^{-2}$$

Example Suppose that the mean time to failure of an electrical component is 5 years. What is the probability that the component is still functioning after 8 years?

$$P(T > 8) = e^{-\frac{8}{5}} = 0.2$$

Suppose now that we have three such components. What is theprobability that at least 2 such components are still functioning after 8 years?

here we have a binomial with parameters $n = 3, p = 0.2. P(X \geq 2) =$

$\binom{3}{2}(0.2)^2\,0.8 + (0.2)^3$

The memoryless property.

The exponential distribution has the property that the probability of survival past a time $t_0 + t$ given that it has survived past a time t_0 is the same as the probability that it will survive past tiem t independently of t_0. mathematically,

$$\begin{aligned} P(T \geq t_0 + t | T \geq t_0) &= \frac{P(T \geq t_0 + t, T \geq t_0)}{P(T \geq t_0)} \\ &= \frac{P(T \geq t_0 + t)}{P(T \geq t_0)} \\ &= \frac{e^{-\lambda(t_0+t)}}{e^{-\lambda t_0}} \\ &= e^{-\lambda t} = P(T \geq t) \end{aligned}$$

The implication of this proprty is that components whose lifetime distribution can be modelled by an exponential distribution do not have to be replaced regularly. A light bulb for example has to be replaced only when it burns out. The big issue is then, how do we know if a lifetime distribution can be modelled by an exponential distribution? This question can be answered using statistical techiques as described in Chapter 6 section 6.10 p.277.

The Gamma distribution generalizes the exponential.

Define the gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \alpha > 0$$

We note $\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \Gamma(n+1) = n!$ for positive integers n.

The continuous random variable X has a gamma distribution with parameters α, β if its density is given by

$$f(x; \alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} & x^{\alpha-1} e^{-x/\beta} x > 0, \alpha > 0, \beta > 0 \\ 0 & otherwise \end{cases}$$

The mean and variance of the gamma distribution are

$$\mu = \alpha\beta, \sigma^2 = \alpha\beta^2$$

Here, α describes the specified number of poisson events which must occur and β is the mean time between failures.

In this formulation, the mean and variance of the exponential distribution are

$$\mu = \beta, \sigma^2 = \beta^2$$

Example Suppose that we receive on average 5 e-mails per minute. What is the probability that we will have to wait at most 1 minute before two calls arrive?

Here, $\beta = 1/5.\alpha = 2$. Integrating by parts, we get

$$\begin{aligned} P(X \leq 1) &= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_0^1 x^{\alpha-1} e^{-x/\beta} dx \\ &= 1 - e^{-5} - 5e^{-5} \end{aligned}$$

1.8 Chi-squared distribution

The chi-squared distribution is a special case of the gamma with $\alpha = \nu/2, \beta = 2$.The mean and variance of the chi-squared distribution are

$$\mu = \nu, \sigma^2 = 2\nu$$

Table A.5 p.441 provides critical values of the chi-squared distribution.

Proposition If $X \sim N\left(\mu, \sigma^2\right)$, then $\left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi_1^2$

Example If $X \sim N\left(\mu, \sigma^2\right)$, we can calculate from Table A.5 p.441 and p.442

$$P\left(0.455 < \left(\frac{X-\mu}{\sigma}\right)^2 < 5.024\right) = 0.50 - 0.025 = 0.4750$$